MIDTERM: RINGS AND MODULES

Date: **24th February 2026**
The Total points is **108** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity** unless specified otherwise.

(1) (10 points) Let R be an integral domain. Which of the following are an integral domain? No justification needed.
  (a) $R \times R$ with component-wise addition and multiplication.
  (b) $R[X]$, the polynomial ring over R.
  (c) Any subring of R containing unity of R.
  (d) Any ring A such that R is a subring of A.

(2) (6+4=10 points) Define Euclidean domain, Principal Ideal domain and Unique factorization domain. State true or false.
  (a) Every Euclidean domain is a Principal Ideal Domain.
  (b) Every Unique Factorization Domain is a Principal Ideal Domain.
  (c) A prime element in an integral domain is irreducible.
  (d) Two distinct nonzero prime ideals in any integral domain are co-maximal ideals.

(3) (8+8=16 points) Prove or disprove. The following two rings are isomorphic.
  (a) $\mathbb{Q}[x]$ and $\mathbb{Q}[x,y]/(xy-1)$
  (b) $\mathbb{Z}[x]/(2x-1)$ and $\mathbb{Z}[\frac{1}{2}]$

(4) (3+21=24 points) Define reduced ring. Give examples of three rings (R_1, R_2 and R_3 such that R_1 is a not a reduced ring, R_2 is a reduced ring which is not an integral domain and R_3 is an integral domain such that as groups under addition $(R_1,+)$, $(R_2,+)$ and $(R_3,+)$ are all isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

(5) (4+11=15 points) Define idempotent elements of a ring. Show that the rings $Q[X]/(X^2-1)$ and $Q \times Q$ are isomorphic.

(6) (6+12=18 points) Define nilradical and Jacobson radical of a ring. Let $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$ be an integer with p_i distinct primes. For the ring $(\mathbb{Z}/n\mathbb{Z})[x]$ compute the nilradical and the Jacobson radical.

(7) (4+11=15 points) Define irreducible and prime elements. Show that in the ring $\mathbb{Z}[\sqrt{-5}]$, 2 is irreducible but not prime.